Dynamic behavior of the critical 2^∞ attractor and characterization of chaotic attractors at bifurcations in a one-dimensional map

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Dynamic behavior of the critical 2^∞ attractor at the accumulation point of period doubling in the one-dimensional map $x_{n+1}=x_n\exp[A(1-x_n)]$ is studied by the sum of the local expansion rates $S_n(x_1)$ of nearby orbits. The variance $\langle [S_n(x)]^2\rangle$ exhibits self-similar structure. Critical bifurcations such as band merging, crisis, and intermittency are studied in terms of $\sigma_n(q)$—the variance of fluctuations of the coarse-grained local expansion rates of nearby orbits.

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Even though chaotic attractors look very complicated, they contain order. The fractal dimension and Lyapunov exponent describe only their global scaling structures. The structures of chaotic attractors are built up by the expanding and folding processes of chaotic orbits. Both theoretically and experimentally, it has been indicated that the structures of chaotic attractors can be characterized by the generalized dimensions $D(q)$ and the singularity spectra $f(\alpha)$ of the natural invariant measures [1–4]. The most important concept of chaos would be orbital instability due to the exponential separation of nearby orbits in phase space, which leads to a positive Lyapunov exponent Λ^∞ for chaotic orbit. Orbital instability brings about sensitive dependence on its initial point and mixing of orbits leading to loss of memory in chaos.

Recently, the features associated with the fluctuation of local expansion rates of various attractors have been studied [5–9]. Chaotic attractors have various local structures depending on their routes of onset and evolution. Particularly, bifurcations of chaos change the structures of chaotic attractors drastically, so that the local structures produce singularity of the natural invariant measure and create the coherent large fluctuations of coarse-grained expansion rates Λ. Such a local structure is called a singular local structure. The mean value of $\Lambda_n(x_1)$ that is equal to Λ^∞ cannot describe singular local structures. The local structures produce coherent large fluctuations of coarse-grained local expansion rates $\Lambda_n(x_1)$ around the positive Lyapunov exponent. Therefore, studying the fluctuations of $\Lambda_n(x_1)$ around Λ^∞ is very important.

In this Brief Report we study the singular local structures of chaotic attractors of the one-dimensional map [10,11]

$$x_{n+1}=x_n\exp[A(1-x_n)] \quad (1)$$

at various bifurcation points, and show that they can be characterized by the dynamic structure function $\sigma_n(q)$, the variance of fluctuations of $\Lambda_n(x_1)$ around $\Lambda_n(q)$, $(-\infty<q<\infty)$. We have calculated $\sigma_n(q)$ for parametric values far away and near the bifurcations, such as band merging, intermittency, and crisis. For all the chaotic attractors of (1) the $\sigma_n(q)$ versus q plot exhibits a peak at $q=q_\alpha$. We show that additional peaks, however, occur only for the attractors just before and after the bifurcations.

For an orbit $\{x_n\}$ $\{n=1,2,\ldots\}$ of a one-dimensional map $x_{n+1}=f(x_n)$, the one-dimensional local expansion rates $\Lambda(x_1)$ are given by

$$\Lambda(x_n)=\ln|df(x_n)/dx_n| . \quad (2)$$

In map (1), as the control parameter A is varied we find several points at which chaotic attractors make drastic changes. The map exhibits a cascade of period doubling to chaos when A is increased from a small value. This cascade accumulates at the critical point $A=A_c\approx 2.692\,368\,900\,3\ldots$ at which the attractor becomes a critical 2^∞ attractor. For this attractor the sum of the local expansion rates of nearby orbits over time n is given by

$$S_n(x_1)=\sum_{t=1}^{n}\Lambda(x_t),\quad n=1,2,\ldots . \quad (3)$$

For (1) S_n takes the form

$$S_n(x_1)=\sum_{t=1}^{n}\ln|(1-Ax_t)\exp[A(1-x_t)]| . \quad (4)$$

The sum $S_n(x_1)$ fluctuates with n, depending upon the initial value x_1. The coarse-grained local expansion rate is given by

$$\Lambda_n(x_1)=S_n(x_1)/n=(1/n)\sum_{t=1}^{n}\Lambda(x_t) . \quad (5)$$

The rate $\Lambda_n(x_1)$ takes different values between a maximum value $\Lambda_{n,\max}$ and minimum value $\Lambda_{n,\min}$ depending on the initial point x_1. As $n\to\infty$, $\Lambda_n(x_1)$ converges to a positive Lyapunov exponent Λ^∞, which is independent of x_1 for almost all values of x_1 within the basin of attraction of the attractor. For a chaotic orbit $S_n(x_1)$ grows with n for large n. However, for the critical 2^∞ attractor at $A=A_c$ Λ^∞ is zero, indicating that the attractor has no mixing and $S_n(x_1)$ grows, at most, less than linear with n. We consider the variance

$$\langle [S_n(x)-n\Lambda^\infty]^2\rangle=(1/N)\sum_{m=1}^{N}[S_n(x_m)-n\Lambda^\infty]^2 . \quad (6)$$

For $A=A_c$, the Lyapunov number Λ^∞ is zero and the variance has an interesting behavior. Figure 1 shows the variance as a function of $\log_2 n$, where N is taken as 2^{11}. The plot shows a fascinating temporal structure. The m^{th} block lying between $\log_2 n=m$ and $\log_2 n=m+1$, $m=0,1,2,\ldots$ consists of 2^{m+1} points. As m increases the block exhibits a self-similar pattern. Thus the initial memory lasts infinitely without mixing and creates the self-similar temporal structure.

Next, we study the chaotic attractors of (1) near various bifurcations in terms of the q-phase transitions. The probability density for $\Lambda_n(x_1)$ to take a value around Λ is given by

$$P(\Lambda;n)=\langle \delta[\Lambda_n(x_1)-\Lambda]\rangle ,$$

where $\delta(f)$ is the delta function of f and $\langle\ \rangle$ denotes the long time average. The probability density function takes the scaling form [12–15]

$$P(\Lambda;n)=\exp\{-n\Psi(\Lambda)\}P(\Lambda^\infty;n) \quad (7)$$

for $n\to\infty$, and $\Psi(\Lambda)$ is a concave function of Λ and takes minimum value zero at $\Lambda=\Lambda^\infty$. To describe the fluctuations of the local expansion rates consider the partition function

$$Z_n(q)=\langle \exp[-n(q-1)\Lambda_n(x_1)]\rangle \quad (-\infty<q<\infty) , \quad (8)$$

the temporal scaling exponent

$$\phi_n(q)=-(1/n)\ln Z_n(q) , \quad (9)$$

and its derivatives

$$\Lambda_n(q)=d\phi_n(q)/dq =[1/Z_n(q)] \times\langle \Lambda_n(x_1)\exp[-n(q-1)\Lambda_n(x_1)]\rangle , \quad (10)$$

$$\sigma_n(q)=-d\Lambda_n(q)/dq =[n/Z_n(q)]\langle [\Lambda_n(x_1)-\Lambda_n(q)]^2 \times\exp[-n(q-1)\Lambda_n(x_1)]\rangle . \quad (11)$$

where $\langle\ \rangle$ denotes the long time average

$$\langle G(x_1)\rangle=\lim_{N\to\infty}(1/N)\sum_{i=1}^{N}G(x_i) .$$

$\sigma_n(q)$ is the variance of fluctuations of $\Lambda_n(x_1)$ around $\Lambda_n(q)$. For $q=1$ $\phi_n(1)$ become zero and $\Lambda_n(1)=\Lambda^\infty$. Further, $\Lambda_n(\infty)=\Lambda_{n,\min}$ and $\Lambda_n(-\infty)=\Lambda_{n,\max}$. Therefore, $\Lambda_n(q)$ and $\sigma_n(q)$ with $q>1$ and $q<1$ can explicitly describe negative and positive large fluctuations of $\Lambda_n(x_1)$, respectively. The dynamical structure function $\sigma_n(q)$ is computed numerically for the critical attractors of the map (1).

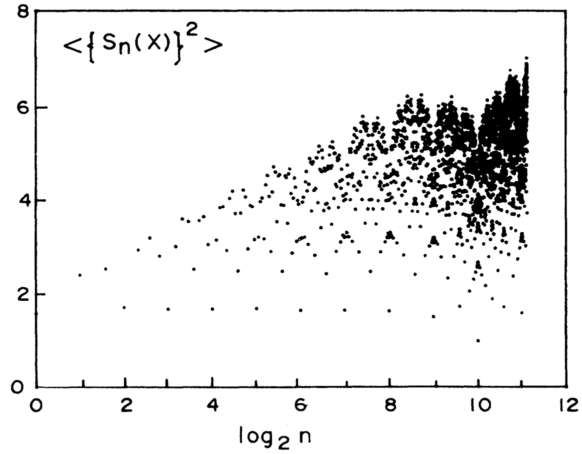

FIG. 1. Variance $\langle [S_n(x)]^2\rangle$ versus $\log_2 n$ for the critical 2^∞ attractor of (1), where $A=A_C\approx 2.692\,368\,900\,3$ and $N=2000$.

The map (1) exhibits band merging bifurcation as A is increased from A_c. Band merging is found to occur at $A=A_b\approx 2.833\,155\,05\ldots$. Figure 2(a) shows the dynamic structure function $\sigma_n(q)$ at $A=2.8$, far from the band merging bifurcation point $A=A_b$. The q-weighted variance $\sigma_n(q)$ has only one peak at $q=q_\alpha=1.3$. At $A=2.832$, just before the merging, $\sigma_n(q)$ exhibits two

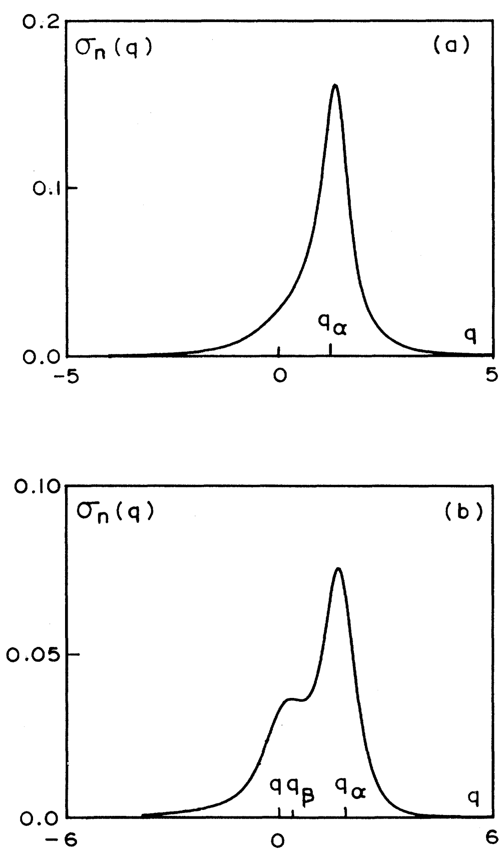

FIG. 2. Dynamic structure function for $A=2.8$ (a) far from the band merging bifurcation and for $A=2.832$ (b) just before the band merging, where $n=100$ and $N=8000$.

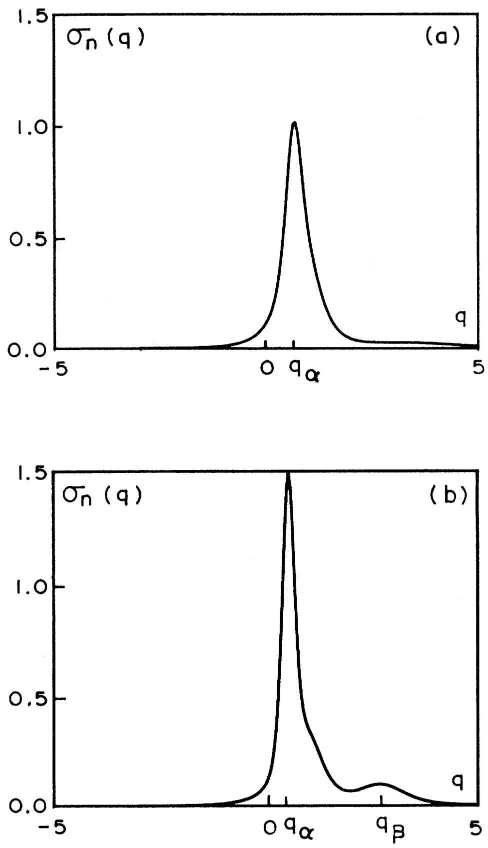

FIG. 3. $\sigma_n(q)$ versus q for (a) $A=3.1024$, far from the crisis, and (b) for $A=3.102438$, just before crisis, where $n=21$ and $N=8000$.

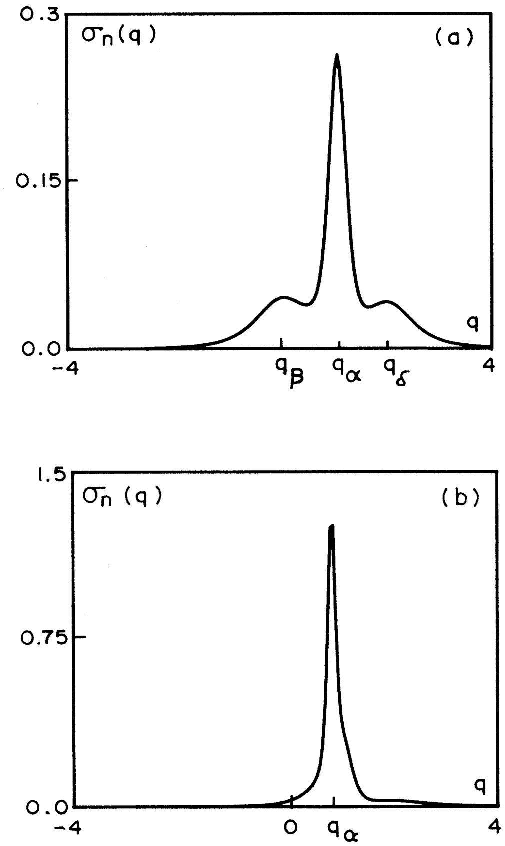

FIG. 4. $\sigma_n(q)$ versus q with $n=100$ and $N=8000$. (a) In the intermittent chaos region, where $A=3.62767$. (b) Far after the intermittent chaos, where $A=3.627676$.

peaks at $q_\alpha=1.65$ and $q_\beta=0.35$, as shown in Fig. 2(b). Two peaks are observed at the band merging and also just after the band merging.

Next, we consider the crisis, the sudden destruction of a chaotic attractor. A chaotic attractor exists for $A < A_d \approx 3.1024395\ldots$. At $A = A_d$ the chaotic attractor suddenly disappears and a period-3 window is found. The dynamic structure functions far from and just before the crisis are depicted in Figs. 3(a) and 3(b), respectively, where $n=21$ and $N=8000$. At $A=3.1024$, far from the crisis, $\sigma_n(q)$ has a single peak at $q=q_\alpha=0.7$. Just before the crisis two peaks occur at $q=q_\alpha=0.55$ and $q_\beta=2.7$, as shown in Fig. 3(b).

Finally, we consider the intermittency region. In map (1) type III intermittent chaos is observed when A is increased from 3.6276701. At $A_I \approx 3.6276720$ fully developed chaotic motion is found. Figure 4(a) shows $\sigma_n(q)$ versus q for $A=3.62767$, just below A_I. $\sigma_n(q)$ has three peaks at $q_\alpha=2$, $q_\delta=1.1$, and $q_\beta=0.075$. Figure 4(b) shows the result for $A=3.627676$, far after intermittency; $\sigma_n(q)$ has only one peak at $q=q_\alpha=0.9$.

In summary, it is shown that at the accumulation of period doubling phenomena the variance $\langle [S_n(x)-n\Lambda^\infty]^2\rangle$ exhibits self-similar temporal structure. Further, the large fluctuations of the coarse-grained local expansion rates of nearby orbits near certain critical bifurcations is also studied. The chaotic attractors just near the bifurcations such as band merging, sudden destruction, and intermittency have singular local structures that produce large fluctuations of the coarse-grained local expansion rates Λ, and consequently, additional peaks are found to occur in the $\sigma_n(q)$ versus q plot. Thus, it turns out that $\sigma_n(q)$ is useful for characterizing the chaotic attractors at their bifurcation points.

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